NOTE ON CERTAIN FORMULAS FOR THE DESIGN OF REENFORCED CONCRETE BEAMS.

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IN chapter II of the book on Reenforced Concrete, by A. W. Buel and C. S. Hill, some formulas and deductions are given which are incorrect. It is the object of this paper to point out the mistakes in these formulas and deductions, and to illustrate the errors involved by numerical examples.

Before proceeding to a consideration of the formulas, it will be well to notice the assumptions on which the formulas are based. It is not proposed to enter into any discussion concerning the correctness of these assumptions:

1. The strain of any fiber is directly proportional to the distance of that fiber from the neutral axis of the beam.

2. The stress in any fiber is proportional to its distance from the neutral axis.

3. Any variation of stress or strain near the reenforcement is neglected.

Referring to figure 1, the author uses the following notation:

$M =$ bending moment in inch-pounds.
$l =$ length of beam in inches, center to center of supports.
$h =$ depth of beam in inches, out to out concrete.
$b =$ breadth of beam in inches.
$A =$ total area of cross-section of beam $= hb$.
$A_s =$ area of steel reenforcement in tension.
$A_s^1 =$ area of steel reenforcement in compression.
$A_c = A - (A_s + A_s^1) =$ area of concrete.
$x =$ distance from neutral axis to outer compression fiber of concrete.
$y =$ distance from neutral axis to outer tension fiber of concrete.
$u =$ distance from neutral axis to outer compression fiber of steel.
$z =$ distance from neutral axis to outer tension fiber of steel.
$t =$ distance from neutral axis to center of steel sections in compression.
$v =$ distance from neutral axis to center of steel sections in tension.
$y = z + d$; $x = u + d^1$; $h = x + y$; $h^1 = u + z$; $h^{11} = t + v$; $y = v + d_1$; $x = t + d_1^1$.

I = moment of inertia of reenforced beam about its neutral axis $= I_s + I_c$.

I_s = moment of inertia of steel about neutral axis of beam.

I_c = moment of inertia of concrete about neutral axis of beam.

E_s = modulus of elasticity of steel.

E_c = modulus of elasticity of concrete.

$e = \frac{E_s}{E_c}$.

f_s = maximum intensity of tensile stress in steel.

f_s^1 = maximum intensity of compressive stress in steel.

f_c = maximum intensity of tensile stress in concrete.

f_c^1 = maximum intensity of compressive stress in concrete.

L_s = load sustained by steel.

L_c = load sustained by concrete.

s = proportion of load sustained by steel.

c = proportion of load sustained by concrete.

The steel reenforcement may consist of one or any number of bars, of any cross-section on both sides of the neutral axis or on the tensile side only.

After stating a number of formulas relating to the bending moment on beams, which have no bearing on the subject of this paper, the author gives these formulas without any deductions:

(1) $$y = \left\{ \frac{bh^2}{2} + \left[A_s d_1{}^2 + A_s{}^1 (h - d_1{}^1)^2 \right] (e-1) \right\} \div \left[A + (A_s + A_s{}^1)(e-1) \right]$$

(2) $$x = h - y$$

(3) $$z = y - d$$

(4) $$u = x - d^1$$

(5) $v = y - d_1$ where $d_1 = d +$ radius of steel.

(6) $t = x - d_1{}^1$ where $d_1{}^1 = d^1 +$ radius of steel.

(7) $I_s = A_s v^2 + A^1{}_s t^2 +$ (the moments of inertia of the steel bars about their individual neutral axes, which may be neglected for small rods without appreciable error).

(8) $$I_c = \frac{bh^3}{12} - \left(A_s v^2 + A_s{}^1 t^2 \right)$$

The distribution of the load between the concrete and the steel will be:

(*9*) $\frac{L_c}{L_s}=\frac{E_cI_c}{E_sI_s}=\frac{I_c}{eI_s}=g$

(*10*) $c=\frac{g}{1+g}$ and $s=\frac{1}{1+g}$

(*11*) $M=\frac{I_sf_s}{z\,s}$

(*12*) $M=\frac{I_sf_s^1}{u\,s}$

(*13*) $M=\frac{I_cf_c}{y\,c}$

(*14*) $M=\frac{I_cf_c^1}{x\,c}$

If $f_c=0$, that is, if the concrete does not resist tensile stresses—the condition that exists after the concrete has failed in tension, but before the ultimate resistance of the reenforced beam has been reached—we have: (*15*)

$$x=-\frac{e}{b}\left(A_s+A_s^1\right)+\sqrt{\frac{e^2(A_s+A_s^1)^2}{b^2}+\frac{2e}{b}\left[A_s^1d_1^1+A_s(h''+d_1^1)\right]}$$

(*16*) $I_c=\frac{bx^3}{3}$, nearly, if A_s and A_s^1 are small.

(*17*) $I_s=A_sv_2+A_s^1t^2$, nearly.

If $A_s^1=0$, that is if the steel reenforcement is on the tensile side of the neutral axis only,

(*18*) $x=-\frac{e}{b}A_s+\sqrt{\frac{e^2A_s^2}{b^2}+\frac{2e}{b}A_s(h''+d_1^1)}$

(*19*) $I_c=\frac{bx^3}{3}$, nearly, if A_s is small.

(*20*) $I_s=A_sv^2$, nearly.

Proceeding now to an examination of some of these formulas in detail, a value for y of (*1*) can be found as follows:

If the rods are small, f_s and f_s^1 may be taken as the intensity of stress throughout the entire section of the rods in tension and compression, respectively.

For equilibrium:

The sum of the compressive stresses = the sum of the tensile stresses, or symbolically,

$$F_c = F_t \tag{1}$$

Let C_c and C_t equal the compressive and tensile stresses in a concrete beam of the same dimensions as the reenforced beam.

Let S_c and S_t equal the excess of stress in the steel over that in the same area of concrete in the same position in the beam.

Then (*1*) may be expressed

$$C_c + S_c = C_t + S_t \tag{2}$$

$$\text{Now } C_c = \frac{bxf_c^1}{2} \tag{3}$$

$$S_c = f_s^1 A_s^1 - \left(\frac{x - d_1^1}{x} f_c^1\right) A_s^1 \tag{4}$$

$$C_t = \frac{byf_c}{2} \tag{5}$$

$$S_t = f_s A_s - \left(\frac{y - d_1}{y} f_c\right) A_s \tag{6}$$

Substituting these values in (*2*)

$$\frac{bxf_c^1}{2} + \left[f_s^1 A_s^1 - \left(\frac{x - d_1^1}{x} f_c^1\right) A_s^1\right] = \frac{byf_c}{2} + \left[f_s A_s - \left(\frac{y - d_1}{y} f_c\right) A_s\right] \tag{7}$$

Now, since by our assumption the strain in any fiber is proportional to its distance from the neutral axis, and since $\frac{F}{A} = \frac{\lambda E}{l}$, λ A and l remaining the same for either material at a given position in the beam, $\frac{F}{A}$ must vary as E, A representing any area under consideration.

Now for steel in tension $\frac{F}{A} = f_s$

For steel in compression $\frac{F}{A} = f_s^1$

For concrete in tension $\frac{F}{A} = \frac{y-d_1}{y} f_c$

For concrete in compression $\frac{F}{A} = \frac{x-d_1^1}{x} f_c^1$

at those distances from the neutral axis at which the reenforcing rods are placed.

$$\text{From the foregoing:} \quad \frac{f_s^1}{\frac{x-d_1^1}{x} f_c^1} = \frac{E_s}{E_c}$$

$$\text{or} \quad f_s^1 = \frac{x-d_1^1}{x} f_c^1 e \qquad \text{since} \quad e = \frac{E_s}{E_c}$$

$$\text{Similarly} \quad f_s = \frac{y-d_1}{y} f_c e$$

Substituting these values of f_s^1 and f_s in (*7*) we have:

$$\frac{bxf_c^1}{2} + A_s^1\left[\frac{x-d_1^1}{x} f_c^1 (e-1)\right] = \frac{byf_c}{2} + A_s\left[\frac{y-d_1}{y} f_c(e-1)\right] \qquad (8)$$

Dividing by f_c^1 and remembering that $\frac{f_c}{f_c^1} = \frac{y}{x}$

$$\frac{bx}{2} + A_s^1 \frac{x-d_1^1}{x}\left(e-1\right) = \frac{by}{2} \cdot \frac{y}{x} + A_s \frac{y-d_1}{y} \cdot \frac{y}{x}\left(e-1\right)$$

Or,

$$\frac{bx^2}{2} + A_{s1}\left(x-d_1^1\right)\left(e-1\right) = \frac{by^2}{2} + A_s\left(y-d_1\right)\left(e-1\right) \qquad (9)$$

Substituting for x, its value $h-y$

$$b(h-y)^2 + 2A_s^1(h-y-d_1^1)(e-1) = by^2 + 2A_s(y-d_1)(e-1). \qquad (10)$$

Solving this equation for y gives

$$y = \frac{\frac{bh^2}{2} + [A_s d_1 + A_s^1(h-d_1^1)]\,(e-1)}{A + (A_s + A_s^1)\,(e-1)}$$

which is the correct value of y.

This formula may be deduced very simply in another way from the well-known theorem that, with both stresses and strains proportional to the corresponding distances from the neutral axis, the neutral axis must pass through the center of gravity of a homogeneous section. To bring our non-homogeneous section under this theorem, we must replace the steel by concrete, and also add enough concrete at the same distance from the top and bottom of the beam to take the excess of stress in steel, over that taken by the concrete in its place. (See fig. 2.) The unit stresses in the steel are $f_s^1 = e\frac{x-d_1^1}{x}f_c^1$ and $f_s = e\frac{y-d_1}{y}f_c$. The total stresses taken by the steel are $f_s^1A_s^1 = e\frac{x-d_1^1}{x}f_c^1A_s^1$ and $f_sA_s = e\frac{y-d_1}{y}f_cA_s$.

The unit stresses in the concrete at this same position are

$$\frac{x-d_1^1}{x}f_c^1 \text{ and } \frac{y-d_1}{y}f_c$$

So the stresses taken by the concrete replacing the area of the steel will be

$$\frac{x-d_1^1}{x}f_c^1A_s^1 \text{ and } \frac{y-d_1}{y}f_cA_s$$

Hence, denoting by A_c^1 and A_c the *additional* area of concrete in compression and tension, we have

$$A_c^1 = \frac{e\frac{x-d_1^1}{x}f_c^1A_s^1 - \frac{x-d_1^1}{x}f_c^1A_s^1}{\frac{x-d_1^1}{x}f_c^1} = \left(e-1\right)A_s^1$$

and $A_c = (e-1)A_s$. Taking moments about the bottom of the section, we have, for the position of the center of gravity of this area,

$$y = \frac{\frac{bh^2}{2} + \left(e-1\right)A_s^1\left(h-d_1^1\right) + \left(e-1\right)A_sd_1}{bh+(e-1)A_s^1+(e-1)A_s}$$

or,

$$y = \frac{\frac{bh^2}{2} + \left[A_sd_1 + A_s^1\left(h^1-d_1^1\right)\right](e-1)}{A+(A_s+A_s^1)(e-1)}$$

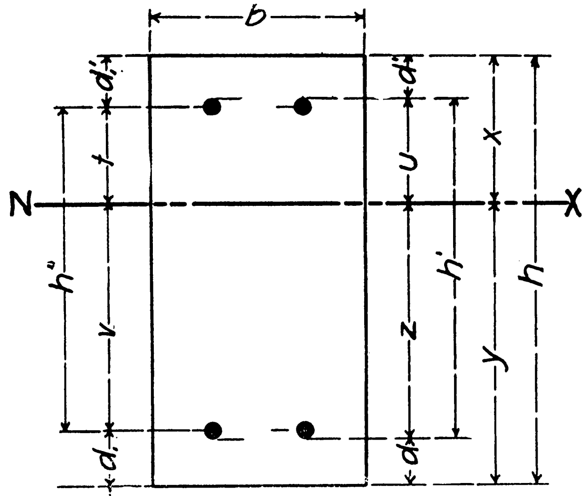

FIG. 1.

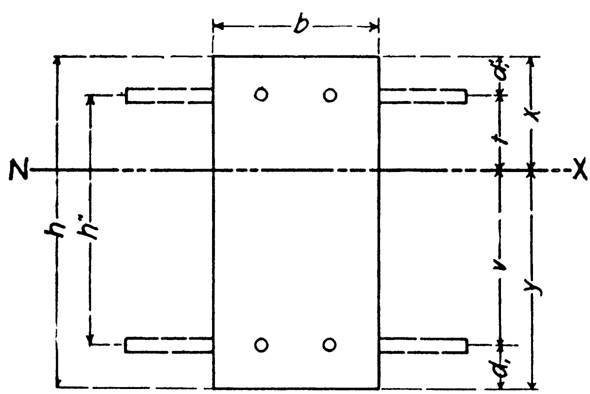

FIG. 2.

Comparing this with the author's formula (*1*), we notice that the terms d_1 and $h^1-d_1^1$ are raised to the second power, and that the sign of addition between the two main terms should be replaced by a sign of division. The latter is a mere misprint, as is evident from an illustrative example worked out on the succeeding page in the book. But the exponent is retained in this same example, showing that it is designed to be there. The numerical significance of the mistake may be shown by an example.

Consider a beam 4 inches wide and 16 inches deep. Let there be a rod 1 inch in diameter, so placed that its center is $1\frac{1}{2}$ inches

from the top of the beam, and a similar rod at the same distance from the bottom. Then $b=4$, $h=16$, $d_1=d_1^1=1\frac{1}{2}$, $A_s=A_s^1=.7854$ sq. in. Assume $E_s=29{,}000{,}000$ and $E_c=2{,}900{,}000$. Then $e=10$ Substituting these values in the author's formula (1), we have:

$$y=\frac{4\times\frac{16^2}{2}+\left[.7854(1.5)^2+.7854(14.5)^2\right]9}{64+(1.5708)9}$$

$=25.8$ inches. Substituting these same values in our new formula, we have

$$y=\frac{4+\frac{16^2}{2}+\left[.7854\times 1.5+.7854\times 14.5\right]9}{64+(1.5708)\times 9}=8''$$

which is the correct result, since our beam was assumed symmetrical. Probably the reason that the author did not discover such an obvious mistake is because he assumed in his numerical example a beam having no reenforcement on the upper side. And it is in the square of the large term $h-d_1^1$ that the greatest error appears. Now, in the case of a built-in beam, reenforcement would be put in the top of the beam for at least part of its length.

The next formula that requires comment is formula (*8*)

$$I_c=\frac{bh^3}{12}-\left(A_sv^2+A_s^1t^2\right)$$

This should be $I_c=\frac{bh^3}{12}+bh\left(\frac{h}{2}-y\right)^2-\left(A_sv^2+A_s^1t^2\right)$

That is, in order to subtract one moment of inertia from another we must have them about the same axis. From the moment of inertia of the entire area about its center of gravity the author subtracts the moment of inertia of the steel about the neutral axis of the beam. This is correct only when these two coincide. And this is not generally the case. The extra term may or may not be a negligible quantity.

If we omit the rod in the upper part of the beam of our preceding example, we have $b=4''$, $h=16''$, $A_s=.7854$ sq. in.

$A_s^1=o$, $d_1=1\frac{1}{2}''$ and $e=10$.

$$y=\frac{\frac{4\times 16^2}{2}+\left[.7854\times 1.5\times 9\right]}{64+.7854\times 9}=7.37''$$

$v=y-d_1=7.37-1.5=5.87''$

whence, using the author's formula,

$$I_c = \frac{4\times 16^3}{12} - .7854\times(5.87)^2 = 1365 - 27 = 1338$$

$$\frac{h}{2} - y = 8 - 7.37 = .63''$$

Our added term is then $bh\left(\frac{h}{2} - y\right)^2 = 64\times .63^2 = 25$, which shows that the missing term is, in certain cases at least, about as important as the second term of the formula given.

The next formula to which we direct attention is formula (*15*). We will apply the second method used in deducing formula (*1*). Now, however, we shall take moments about the top of the beam instead of the bottom.

$A_c{}^1$ = the area to be added to take the excess of stress in steel in compression $= (e-1)A_s{}^1$.

A_c = the area to be added to replace the steel in tension $= eA_s$.

$$eA_s + \left(h'' + d_1{}^1\right) + \left(e-1\right)A_s{}^1 d_1{}^1 + \frac{bx^2}{2} = x\left[eA_s + \left(e-1\right)A_s{}^1 + bx\right]$$

$$x^2 + \frac{2[eA_s + (e-1)A_s{}^1]x}{b} = \frac{2eA_s(h'' + d_1{}^1) + 2(e-1)A_s{}^1 d_1{}^1}{b}$$

whence

$$x = -\frac{e(A_s + A_s{}^1) - A_s{}^1}{b}$$

$$+\sqrt{\frac{[e(A_s + A_s{}^1) - A_s{}^1]^2}{b^2} + \frac{2e[A_s(h'' + d_1{}^1) + A_s{}^1 d_1{}^1] - 2A_s{}^1 d_1{}^1}{b}}$$

The difference here is not large unless $A_s{}^1$ is large. And generally if the concrete did not take tension, all the reenforcing would be on the lower or tensile side of the beam. But in formula (*1*) the author multiplies his areas by $(e-1)$; so, to be consistent, he should do the same here for the area in compression.

We have not discovered any further errors in this set of formulas. There is, however, an important error in their application to which we wish to call attention. The author uses this example:

"What is the safe bending moment for a reenforced concrete beam 16 inches deep, 4 inches wide, having one steel rod $\frac{7}{8}$ inch in diameter on the tensile side only, so placed that the center of

the rod is $1\frac{1}{2}$ inches above the bottom of the concrete? The resistance of concrete to tensile stresses is neglected. $f_c^1=700$, $E_c=3{,}600{,}000$, $E_s=29{,}000{,}000$."

Following is the author's solution of the problem:

"From equation (*18*) we have

$$x = -\frac{8.06}{4.0}\times 0.6+\sqrt{\frac{64.89\times 0.36}{16}+\frac{16.11}{4}0.6\times 14.5}$$

$$= -1.209+\sqrt{1.46+35.04}=4.833''$$

"From equation (*19*) we get

$$I_c=\frac{4\times 112.89}{3}=150.52$$

and from equation (*20*)

$$I_s=93.45\times 0.6=56.07.$$

"Since all the tensile stresses are carried by the steel, and all the compressive stresses by the concrete, and since the tensile stresses must balance the compressive stresses to fulfil the conditions of stability, we have $c=s=0.5$, and from equation (*14*) we obtain

$$M=\frac{150.52\times 700}{4.833\times 0.5}=43{,}601.9 \text{ inch-pounds.''}$$

This solution is correct up to the words, "we have $c=s=0.5$."

This conclusion does not in general follow from the statement preceding it. It will follow when $v=\frac{2}{3}x$, and only then; for, if the stresses in tension and compression are equal, their lever arms must be equal in order to produce equal moments. The tensile stress is evidently applied at a distance of v from the neutral axis. The compressive stress increases from 0 at the neutral axis to f_c^1 at a distance x from the axis. So its point of application must be at a distance of $\frac{2}{3}x$ from the neutral axis. This condition does not obtain in this case.

Further, we should obtain the same moment from the use of equation (*11*) as from equation (*14*).

To use equation (*11*) we must compute f_s and z (=v approximately)

$$v = h - x - d_1 = 16 - 4.833 - 1.5 = 9.667''$$

$$f_s = e\frac{v}{x} f_c^1 = 8.06 \times \frac{9.667}{4.833} \times 700 = 11{,}284.$$

Substituting these values, and s=0.5 in formula (*11*) we have

$$M = \frac{56.07 \times 11284}{9.667 \times 0.5} = 130{,}875 \text{ inch-pounds.}$$

The correct solution is as follows:

From (*9*) $g = \frac{I_c}{eI_s} = \frac{150.52}{8.06 \times 56.07} = .333.$

From (*10*) $c = \frac{g}{1 \times g} = \frac{.333}{1.333} = 0.25$ and $s = \frac{1}{1.333} = 0.75.$

Substituting these values in (*14*) and (*11*), respectively, we get from (*14*): $M = \frac{150.52 \times 700}{4.833 \times .25} = 87{,}203.$

From (*11*), $\frac{56.07 \times 11284}{9.667 \times .75} = 87{,}250;$

which are the same, except for the slight difference due to decimals.